Mathematical Background for Global Robot Localization

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Abstract

This document presents the mathematical background for a global localization method for mobile robots operating in a known occupancy map. The method performs global pose estimation using lidar observations and a precomputed Euclidean distance transform (EDT) derived from a reference occupancy grid. A probabilistic observation model is used to derive a negative log likelihood objective function over robot pose. A coarse-to-fine search procedure is employed to identify candidate pose hypotheses, followed by local refinement and covariance estimation via finite-difference Hessian approximation.

The intent of this document is to provide engineering background for the accompanying ROS2 package implementation.

1 Introduction

We consider the problem of estimating the planar robot pose

$$q = [x, y, \theta]^T \tag{1}$$

from a set of lidar observations and a known occupancy map.

The method described here is intended primarily as a robust global localization and relocalization tool for ROS2-based mobile robots. The intended use case is:

- A static occupancy map has previously been generated.
- The robot is stationary during global localization.
- A set of lidar scans are acquired.
- A global search over pose is performed.
- One or more candidate pose hypotheses are returned.

The implementation is particularly motivated by robot testing when the initial pose (/initialpose) needs to be set. While (currently) it is assumed that the most likely pose is the solution, future work will help the user detect and resolve multiple-solution cases. For example, for the initialization of robot navigation problem, one could use rviz to select the desired solution from the candidates returned by this package.

2 Notation and Coordinate Frames

We assume the following coordinate frames:

- $\mathcal{F}_m$: map frame
- $\mathcal{F}_b$: robot base frame
- $\mathcal{F}_l$: lidar frame

A static transform between the robot base and lidar frame is assumed known. Let the robot pose in the map frame be:

$$q = [x, y, \theta]^T \tag{2}$$

A lidar point observed in the lidar frame is denoted:

$$p_i^{(l)} = [x_i^{(l)}, y_i^{(l)}]^T \tag{3}$$

After application of the base-to-lidar transform and robot pose transform, the point in the map frame is:

$$r_i(q) = T(q)\, p_i \tag{4}$$

where $T(q)$ denotes the rigid planar transformation parameterized by pose q.

3 Reference Occupancy Map and Distance Transform

We assume a known occupancy map represented as a 2D occupancy grid. Occupied cells are used to construct a Euclidean distance transform (EDT):

$$D_m(r) \tag{5}$$

which returns the Euclidean distance from point r to the nearest occupied cell in the map. The EDT is precomputed from the reference occupancy map. The EDT representation is advantageous because:

- It provides a smooth cost function.
- It avoids explicit point correspondences.
- Nearest-obstacle lookup is computationally efficient.

Unknown occupancy regions are not treated as occupied. Furthermore, maps are padded by empty (or unknown) cells to prevent sharp EDT functions due to lidar reflection points at the edge of the occupancy map.

4 Observation Model

Consider a lidar endpoint transformed into the map frame:

$$r_i(q) \tag{6}$$

The distance to the nearest occupied cell is:

$$d_i(q) = D_m(r_i(q)) \tag{7}$$

A common likelihood-field approximation models the probability of a lidar observation as:

$$p(z_i \mid q, m) \propto \exp\left(-\frac{d_i(q)^2}{2\sigma_i^2}\right) \tag{8}$$

where:

- m denotes the map
- z_i denotes the ith lidar observation
- σ_i^2 denotes the effective observation variance

Assuming conditional independence between observations:

$$p(z \mid q, m) \propto \prod_i \exp\left(-\frac{d_i(q)^2}{2\sigma_i^2}\right) \tag{9}$$

Taking the negative log likelihood yields:

$$J(q) = \frac{1}{2}\sum_i \frac{d_i(q)^2}{\sigma_i^2} \tag{10}$$

up to additive constants independent of pose.

5 Effective Observation Variance

The effective observation variance combines multiple independent uncertainty sources. The lidar endpoint uncertainty is modeled approximately as:

$$\sigma_{\text{lidar},i}^2 = \sigma_r^2 + (r_i\sigma_\theta)^2 \tag{11}$$

where:

- σ_r is radial range uncertainty (in meters)
- σ_θ is angular uncertainty (in radians)
- r_i is the measured lidar range

Map uncertainty is represented by:

$$\sigma_{\text{map}}^2 \tag{12}$$

Assuming independent Gaussian uncertainties:

$$\sigma_i^2 = \sigma_{\text{lidar},i}^2 + \sigma_{\text{map}}^2 \tag{13}$$

The resulting variance is treated as a fixed per-observation weighting during optimization.

5.1 Approximations

The model above is intentionally simplified. Several effects are neglected:

- Anisotropic lidar endpoint covariance
- Incidence-angle dependent uncertainty
- Dynamic scan distortion due to robot motion
- Non-Gaussian scattering effects
- Spatial correlation between observations
- Pose-dependent covariance terms

The implementation assumes the robot is stationary during global localization, thereby avoiding scan deskewing and time-varying motion compensation.

6 Coarse-to-Fine Global Search

The objective function $J(q)$ is generally non-convex and may contain multiple local minima. A coarse-to-fine search strategy is therefore employed.

6.1 Coarse Search

A coarse search over (x, y, θ) is performed over a bounded region of the map. The search grid is discretized:

$$q_{ijk} = [x_i, y_j, \theta_k]^T \tag{14}$$

The objective function is evaluated at each candidate pose. Several further computational optimizations are possible:

- Lidar point decimation
- Parallel evaluation
- Multi-resolution map representations

The best candidate hypotheses are retained for refinement.

6.2 Fine Search

A local refinement stage is performed around the best coarse candidate poses. In the current implementation, a grid search over selectable levels of refinement (by a factor of 2 at each refinement) is utilized.

7 Covariance Estimation

Near the optimum pose q^*, the negative log likelihood objective may be approximated locally as quadratic:

$$J(q) \approx J(q^*) + \frac{1}{2}(q - q^*)^T H (q - q^*) \tag{15}$$

where:

$$H = \nabla^2 J(q^*) \tag{16}$$

is the Hessian matrix.

The pose covariance is approximated as [2]:

$$\Sigma_q \approx H^{-1} \tag{17}$$

provided the Hessian is positive definite.

The Hessian is estimated via finite-difference. Let the finite-difference step sizes be:

$$h_x, \quad h_y, \quad h_\theta \tag{18}$$

Typical values are:

- h_x and h_y on the order of one to two map cells
- h_θ on the order of 0.5° to 2°

Let the optimal pose estimate be:

$$q^* = [x^*, y^*, \theta^*]^T \tag{19}$$

and define:

$$J_0 = J(x^*, y^*, \theta^*) \tag{20}$$

The diagonal second derivatives are approximated using centered finite differences.

Diagonal Terms

$$J_{xx} \approx \frac{J(x^* + h_x, y^*, \theta^*) - 2J_0 + J(x^* - h_x, y^*, \theta^*)}{h_x^2} \tag{21}$$

$$J_{yy} \approx \frac{J(x^*, y^* + h_y, \theta^*) - 2J_0 + J(x^*, y^* - h_y, \theta^*)}{h_y^2} \tag{22}$$

$$J_{\theta\theta} \approx \frac{J(x^*, y^*, \theta^* + h_\theta) - 2J_0 + J(x^*, y^*, \theta^* - h_\theta)}{h_\theta^2} \tag{23}$$

Mixed Terms The mixed second derivatives are approximated using centered four-point finite differences.

$$J_{xy} \approx \frac{J(x^* + h_x, y^* + h_y, \theta^*) - J(x^* + h_x, y^* - h_y, \theta^*) - J(x^* - h_x, y^* + h_y, \theta^*) + J(x^* - h_x, y^* - h_y, \theta^*)}{4h_x h_y} \tag{24}$$

$$J_{x\theta} \approx \frac{J(x^* + h_x, y^*, \theta^* + h_\theta) - J(x^* + h_x, y^*, \theta^* - h_\theta) - J(x^* - h_x, y^*, \theta^* + h_\theta) + J(x^* - h_x, y^*, \theta^* - h_\theta)}{4h_x h_\theta} \tag{25}$$

$$J_{y\theta} \approx \frac{J(x^*, y^* + h_y, \theta^* + h_\theta) - J(x^*, y^* + h_y, \theta^* - h_\theta) - J(x^*, y^* - h_y, \theta^* + h_\theta) + J(x^*, y^* - h_y, \theta^* - h_\theta)}{4h_y h_\theta} \tag{26}$$

The Hessian matrix is then assembled as:

$$H = \begin{bmatrix} J_{xx} & J_{xy} & J_{x\theta} \\ J_{xy} & J_{yy} & J_{y\theta} \\ J_{x\theta} & J_{y\theta} & J_{\theta\theta} \end{bmatrix} \tag{27}$$

The resulting covariance estimate is mapped into a ROS2 `geometry_msgs/PoseWithCovarianceStamped` message.

8 Computational Considerations

The dominant computational costs are:

- EDT computation
- Objective evaluation over candidate poses
- Local refinement

The EDT is precomputed once from the reference occupancy map. The global search stage is highly parallelizable. The method scales approximately linearly with:

- number of candidate poses
- number of lidar points
- angular search resolution

9 Future Work

Possible future extensions include:

- 3D lidar support
- GPU acceleration
- map consistency metrics (lost robot!)

10 References

References

[1] S. Thrun, W. Burgard, and D. Fox, *Probabilistic Robotics*, MIT Press, 2005.

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[3] OpenAI, "Algorithm development discussions and various assistance using ChatGPT (GPT-5.4 and GPT-5.5 models)," 2026. `https://chat.openai.com/`